Comment on "The Transition Temperature of the Dilute Interacting Bose Gas" and on "Transition Temperature of a Uniform Bose Gas"

The transition temperature of a uniform, weakly interacting Bose gas has received considerable interest in the past months, with at least three contributions appearing in these pages alone. In [1] Baym *et al.* argued that because of infrared divergences, the shift in temperature due to the weak repulsive interaction relative to a free Bose gas cannot be obtained in perturbation theory, no matter how weak the interaction. To regularize the infrared divergences, they derived a self-consistent equation for the energy spectrum of the elementary excitations and estimated the temperature shift as

$$\frac{T_{\rm c}-T_0}{T_0} = c_0 \left(na^3\right)^{1/3}, \tag{1}$$

with $c_0 \approx 2.9$. Here, $T_0 = (2\pi\hbar^2/mk_{\rm B})[n/\zeta(3/2)]^{2/3}$, with ζ the Riemann zeta function, is the transition temperature of an ideal Bose gas with particle number density n, and a is the s-wave scattering length characterizing the repulsive interaction between the particles.

In [2] Huang addressed the same issue using a virial expansion of the equation of state. He arrived at the conclusion that the shift in $T_{\rm c}$ increases with the square root of the scattering length, rather than linearly with a.

In this Comment we wish to point out that first, contrary to the claim by Baym *et al.* [1], the shift in temperature can be calculated perturbatively, as was done in [3]. And second, that the regime considered by Huang [2] does not describe the Bose-Einstein condensed phase.

Specifically, he confines himself to the regime where the fugacity $z = \exp(\mu/k_{\rm B}T)$ is smaller than unity, corresponding to negative values of the chemical potential μ. This is because the used expression for the pressure as function of z becomes complex for $z > 1$. However, the broken-symmetry phase of a weakly interacting Bose gas is, unlike He-II, characterized by a *positive* value of μ. In particular, at $T = T_{\rm c}$ perturbation theory gives [4] $\mu = 8\pi\hbar^2 an/m$ to linear order in a. Note that in a free Bose gas μ is either negative (in the gas phase) or zero (in the condensed phase).

In [3] the shift in temperature, among other things, was obtained perturbatively by calculating the pressure to the one-loop order in a grand canonical ensemble as in [4], and expanding it in a high-temperature series. The justification of the high-temperature expansion was given *a posteriori* by the observation that the leading term in the expression for the transition temperature is of order $(\mu/a)^{2/3}$, where we recall that μ is an independent variable in a grand canonical ensemble. Since this is large for a small, the high-temperature expansion is consistent with the weak-coupling assumption of perturbation theory.

The emerging infrared divergences were regularized by analytic continuation, frequently used to regularize ultraviolet divergences. Finally, using the equation of state to swap the chemical potential for the particle number density as independent variable—which is more appropriate from an experimental view point—we found that the shift in temperature is given by Eq. (1), with

$$c_0 = -\frac{8}{3}\frac{\zeta(\frac{1}{2})}{\zeta^{1/3}(\frac{3}{2})} \approx 2.83. \tag{2}$$

The estimate of Baym *et al.* is remarkably close to this perturbative result.

Earlier path-integral Monte Carlo simulations [5] restricted to small (up to 216) particle numbers gave a value $c_0 = 0.34 \pm 0.06$ almost an order of magnitude smaller that the perturbative result (2). More recent simulations [6] on larger systems containing of the order of 10^4 particles, gave a value $c_0 = 2.3 \pm 0.25$ in reasonable agreement with Eq. (2).

In closing, we remark that in [3] also a similar high-temperature expansion and regularization of infrared divergences as discussed here was applied to the BCS theory of superconductivity, and shown to reproduce, among other things, the known result for the BCS transition temperature.

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